Journal of Forensic and Investigative Accounting
Volume 15: Issue 1, January–June 2023

An Examination of Non-Linearity in Financial Statement Data Using Topological Tests

Kathleen A. Kaminski
*Eileen A. Hogan**

I. Introduction

Fraud in business has been and continues to be a matter of intense economic concern. In their 2020 Report to the Nations, the Association of Certified Fraud Examiners' (ACFE) studied 2,504 cases of fraud from 125 countries causing total losses of $3.6 billion. While financial statement fraud only accounts for 10% of the cases, it is the costliest, with a median loss of $954,000. The median loss per month for all types of fraud is $8,300; for financial statement fraud, the median loss per month is $39,800 (ACFE, 2020). The U.S. edition of the 2020 PricewaterhouseCoopers' (PwC) Global Economic Crime and Fraud Survey found 56% of U.S. firms experienced fraud within the past 24 months, suffering $6.2 billion in total fraud losses. Accounting fraud is among the top three types of reported fraud, increasing from 21% to 30%. Two out of five U.S. companies do not have formal or documented policies for their overall fraud program and 35% do not regularly test or audit controls. (PwC, 2020). Internal auditors initially discover fraud in 14% of the cases and external auditors in only 4% of the cases (ACFE, 2020). As the ACFE notes, "this is a tiny fraction of the number of frauds committed each year … " (ACFE, 2020, p. 6). The result has been increasing pressure to reduce fraudulent financial reporting and the past 40 years have produced new laws, commission reports, and auditing standards.

There is a strong need for research approaches that enable auditing practitioners to identify potential fraud, particularly focusing on the development of analytical procedures with detection or predictive power. The ACFE 2020 study indicated that 83% of the fraud cases analyzed went through independent audits. With the requirement by the Public Company Accounting Oversight Board (PCAOB) that independent audits of publicly traded companies utilize analytical procedures multiple times throughout the auditing process, determining effective means of increasing the predictability of potential fraud through these procedures should be a driving focus of fraud research.

As early as 1993, Etheridge and Sriram (1993) suggested that chaos theory provides a theoretical framework and techniques to perform additional types of analyses. Following this research, other authors (Kaminski and Wetzel, 2004) proposed that chaos theory better explains financial ratio data. Chaos theory explores the long-term behavior of deterministic non-linear dynamical systems (Kellert, 1993). Chaos theory methodology has at least two tools for measuring the non-linearity of a system. Kaminski and Wetzel (2004) demonstrated the first approach, a metric analysis, employing statistical tests on a set of financial statement data. They found evidence that financial ratios are the result of non-linear chaotic dynamics. As yet unexplored is the value of using the second tool derived from chaos theory, the topological approach. Such analysis may provide additional evidence that financial statement data exhibits non-linear chaotic dynamics.

The present study triangulates the earlier work of Kaminski and Wetzel (2004) by conducting a topological examination of the same data. Following Kaminski and Wetzel (2004), ten financial ratios were computed from quarterly financial statements for 30 matched pairs of U.S. fraud and non-fraud firms over time. Using chaos theory methodology, we performed topological tests to determine the behavior of the time-series data and observe whether there was a difference between firm type. Albeit a more subjective analysis, this may provide additional insight as to the predictability of fraud and whether the pursuit of such should be based on a linear or non-linear model.

The remainder of this article is organized as follows. First, we review the relevant research concerning the use of financial ratios to detect fraud. We follow with an introduction to chaos theory and the terminology and concepts applicable to the ensuing research. The methodology section discusses sample selection, ratios chosen, and the topological tests used for analysis. We then present the results of the empirical examinations followed by the study's conclusions and limitations.

*The authors are, respectively, Assistant Professor of Accounting, University of Louisiana Monroe and Accreditation Coordinator, Black Hills State University.

II. Literature Review

Over the last 40 years, the detection of fraudulent financial reporting has been the subject of much empirical research. Thornhill (1995) posited that analytical procedures (APs) could be a useful tool for identifying fraud (Thornhill, 1995). These procedures involve the analysis of trends, ratios, and reasonableness tests derived from an entity's financial and operating data. Other authors have developed linear models to analyze financial loss and identify fraud. These include the Beneish model (1997, 1999), Altman Z-score model (1968) and Dechow F-score model (2011). Various financial ratios are components of these models.

A complete review of published research on the use of APs to predict financial statement fraud is beyond the scope of this study. Comprehensive literature reviews appear in Nieschwietz et al., 2000; Kaminski et al., 2004; Hogan et al.,2008; Ngai et al., 2011; and Gepp et al., 2018. Below, we chronologically summarize studies that dealt exclusively with the detection or prediction of fraud using financial ratios.

Spathis (2002) developed a model to detect factors associated with fraudulent financial statements. The study matched 38 fraudulent financial statements with non-fraudulent financial statements from Greek manufacturing firms. This research used stepwise-logistic regression analysis, revealing one model comprised of nine ratios, and a second model including Altman's Z-score. Both models correctly classified 83–84% of the firms and indicated two ratios (inventory/sales, total debt/total assets) and the Z-score as potential indicators of fraudulent financial statements.

Kaminski et al. (2004) investigated whether the financial ratios of fraudulent firms differed from those of non-fraudulent firms. Using a matched-pairs design with 79 U.S. firms, they compared 21 ratios for a seven-year period (i.e., the fraud year -/+ 3 years). While 16 ratios were statistically significant, five were significant during the period prior to the fraud year and only three were significant for three time periods. Using discriminant analysis, the fraud firms were misclassified 58–98 percent of the time. Guan et al. (2007) examined 21 financial ratios of 68 fraudulent U.S. firms and found that logit and discriminant analysis were ineffective in predicting the likelihood of fraud.

While some researchers concluded that " … traditional analytical procedures have yielded limited success in identifying fraud" (Hogan et al., 2008, p. 241), others have continued to recommend that financial ratios were an effective tool to detect fraud (Subramanyam and Wild, 2009; Bai et al., 2008). Accordingly, researchers have continued their investigations.

Dalnial et al. (2014a, 2014b) conducted two studies of 65 matched fraudulent and non-fraudulent Malaysian firms between 2000 and 2011. Both studies used logit and stepwise multiple regression to examine eight financial ratios, using firm size as a control variable. The 2014a study tested two models, one comprised of eight ratios and the second including Altman's Z-score. They found that two ratios (total debt/total equity, receivables/revenue) and the Z-score were significant enough to predict fraud. However, their model only correctly classified 49% of the fraudulent firms. "Other results using the model are inconsistent with Spathis (2002)." (Dalnial et al., 2014a, p. 69). In contrast, their 2014b study examined the same eight ratios but for a period of five years, including the fraud year and the four years prior. They found that four ratios (total debt/total equity, receivables/revenue, inventory/total assets and revenue/total assets) were significant predictors of misleading financial statements and correctly classified 55% of the fraudulent firms.

Hasnan et al. (2014), in their examination of issues related to fraudulent financial reporting of 53 Malaysia firms between 1996 and 2007, included a measure of earnings management comprised of several ratios: revenue to total assets, gross property, plant and equipment to total assets and return on assets. They found that the mean was greater for fraud firms for three, two and one year(s) prior to the fraud year but not for years four and five. These findings are inconsistent with those of Kaminski et al. (2004).

Nia (2015) sampled 134 companies listed on the Tehran Stock Exchange during 2008–2014. The study integrated the Beneish M-score model and the Altman's Z-score model, comparing eight ratios of fraudulent and non-fraudulent firms. An independent sample t-test was used to determine the likelihood of fraudulent financial reporting. Significant differences existed between the means for the following ratios: current assets to total assets, inventory to total assets, and revenue to total assets. Once again, these findings are consistent with some studies (Feroz et al., 1991; Dalnial et al., 2014b) and inconsistent with others (Kaminski et al., 2004; Dalnial et al., 2014a). There was not a significant difference for the total debt to total equity, total debt to total assets, working capital to total assets ratios. These findings are inconsistent with Persons (1995), Kaminski et al. (2004). and Spathis (2002).

Aghghaleh et al. (2016) compared the detection power of the Beneish M-score and Dechow F-score models using data from 82 matched pairs of Malaysian firms between 2000 and 2014. These are complicated models that include variables comprised of ratios for different time periods or the change in a given ratio. They found that both models are effective in predicting both fraudulent and non-fraudulent firms with accuracy rates between 73–76%. Type II error rates were between 27–30%. Such findings are inconsistent with other studies (Nia, 2015; Dalnial et al., 2014a; Guan et al., 2007; Kaminski et al., 2004).

Birol (2019) studied 134 firms on the Istanbul Stock Exchange for the period 2010–2014, combining financial ratios with non-financial variables to construct a fraud detection model via logistic regression. Using the same nine ratios as Spathis (2002), they found three ratios to be statistically significant (debt to total assets, receivables to sales and gross profit to total assets) and is inconsistent with Spathis' (2002) results.

Alshehadeh and Atieh (2020) conducted a study of five Jordanian commercial banks for the period 2011–2017, using liquidity, profitability, and solvency ratios. They found no statistically significant impact of APs on detecting material misstatement in the banks' financial statements. These findings are consistent with the studies by Kaminski et al. (2004) yet inconsistent with the studies by Allam and Emad (2012).

Thus, despite continued ongoing investigation of fraudulent financial reporting, various factors and/or models for prediction/detection have resulted in inconclusive and often conflicting results. To put it simply, "The results on effectiveness of financial ratios on fraud detection are mixed." (Aghghaleh et al., 2016, p 57). Perols et al. (2017) has a possible explanation. "Financial statement fraud is characterized by both (1) relative rarity (a.k.a. the needle in the haystack problem) and (2) ... an abundance of explanatory variables proposed in the literature (a.k.a, the curse of data dimensionality problem) ... The curse of data dimensionality is a potential problem in fraud prediction because the number of known fraud cases is small relative to the extensive number of independent variables identified in prior fraud studies. Hence, only a small number of fraud observations are available to identify patterns among a large number of independent variables and fraud. This small number may result in overfitted prediction models that perform poorly when predicting new observations." (Perols et al., 2017, p. 223–224).

Perhaps a different type of examination can provide some insight, one which investigates the nature of the data used to develop the models. Financial statements are the product of a dynamical system which has a feedback loop whereby the output of one period is the starting point for the subsequent period. According to chaos theory, such dynamical systems may be deterministic yet appear random and unpredictable.

III. Chaos Theory

Chaos theory focuses on the behavior of deterministic non-linear dynamical systems (Kellert, 1993). A system is called deterministic when its future states are completely fixed by its current state and its rules of dynamical motion (Casti, 1994).

A well-studied non-linear discrete model exhibiting chaos is the logistic map. May (1976) initially investigated the chaotic properties of the general logistic map:

$$X_{t+1} = a\, X_t\, (1 - X_t)$$

Results are dependent on both the initial starting value X_0 and parameter value a. Holding the initial starting value X_0 constant but changing the parameter value a produces different values. For example, if a is between 0 and 4, then for any X_0 between 0 and 1, all subsequent X_t will be bounded between 0 and 1. Repeated iterations of the equation illustrate the dynamics of the non-linear system and result in different types of behavior. (See Table 1, pg. 10)

Examples of the various behaviors of the logistic equation are given in Table 1. When $a = 2.5$, iteration of the equation quickly converges to the fixed value 0.60. By contrast, when $a = 3.2$, iteration results in an oscillation between two values: 0.513045 and 0.799455. When $a = 3.5$, iteration again results in an oscillation. This time, by X_{36}, the equation produces four values: 0.826941, 0.500884, 0.874997, 0.38282.

When $a = 3.9$, iteration results in values that appear to have no discernible pattern. While an infinite number of values are produced, they are still bounded within the range $0 < X_t < 1$. Yet when $a = 4.2$, by X_6, the values are outside the previously bounded range and quickly approach negative infinity. Thus, despite being a deterministic non-linear equation, prediction is not possible. As demonstrated, both order and chaos may appear within the same deterministic system. Such systems may appear random, but underlying patterns and deterministic principles that are highly sensitive to initial condition actually govern this seemingly apparent randomness.

This theoretical impossibility of prediction results from three characteristics of the iterative process: (1) numbers extend to an infinite number of decimal places and must be rounded off in order to do practical calculations; (2) the outcome has a sensitive dependence upon the chosen parameters; and (3) in the chaotic range, the outcome has an extremely sensitive dependence on initial conditions (SDIC). SDIC means that small initial differences or fluctuations in variables may grow over time into large differences.

The logistic map illustrates the concept of sensitive dependence on parameters. Once again, set $X_0 = .20$ (see Table 1). When $a = 3.9$, iteration results in a seemingly random set of solutions bounded within the range $0 < X_t < 1$. At $a = 3.900001$, the solution set is again infinite and seemingly random and bounded within the same range. Comparison of the two solution sets is similar only for the first 25 iterations or so. By X_{28}, the solution sets begin to diverge. Further iterations sometimes bring the solutions closer together (e.g., see X_{51}, X_{57}) only to become divergent once again (e.g., see X_{54}, X_{65}).

Similarly, the logistic map illustrates the concept of SDIC. Set $a = 3.9$ (see Table 1). When $X_0 = .200000$, iteration results in a seemingly random set of solutions bounded within the range $0 < X_t < 1$. At $X_0 = .200001$, the solution set is again infinite and seemingly random and bounded within the same range. The two solution sets are similar only for the first 20 iterations. The sets then begin to diverge. Some iterations bring the solutions closer together (e.g., see X_{34}, X_{56}) while others cause divergence (e.g., see X_{27}, X_{58}).

Both concepts illustrate that effects may be wildly out of proportion to causes. According to chaos theory, small differences in internal states can result in large differences in later states. In layman's terms, this theory is known as the "butterfly effect." A small cause such as a simple and seemingly insignificant rounding can have vey major consequences, which is the nature of chaos. Small changes in initial conditions produce dramatically different evolutionary outcomes. A chaotic system is inherently unpredictable, not because its solution is seemingly random, but because one is unable to measure its initial state with absolute precision. While deterministic non-linear systems are highly predictable in theory (given infinite precision), they are extremely unpredictable in practice where precision is limited.

Topological Analysis

An important arena for understanding non-linear dynamical systems is phase space, a mathematically constructed conceptual space where each dimension corresponds to one variable of the system. Every point in phase space represents a full description of the system in one of its possible states. Phase space is the space of the possible, containing not just the states that do occur but also those that might have occurred. Parameter space is similar but each dimension corresponds to a different parameter. A point in parameter space specifies the values of all the parameters of the system. For example, the logistic map has one variable X and one parameter a. Phase space is therefore the line of X values from 0 to 1 while parameter space is the line of a values from 0 to 4.

Topological analysis depicts the evolution of the system as the tracing out of a path or trajectory in phase space (Kellert, 1993). One can categorize the possible trajectories according to their shape resulting in a topological taxonomy. By linking topology and dynamical systems, phase space provides a way of turning numbers into pictures, abstracting all the essential information from the system and making a flexible road map to all of its possibilities. One can use the shape to visualize the whole range of behaviors of a system. If one can visualize the shape, one can gain understanding of the system. Traditional time-series and trajectories in phase space are two ways of displaying the same data and gaining a picture of a system's long-term behavior (Gleick, 1987).

When a deterministic non-linear dynamical system is plotted in phase space, the resulting shape is called an attractor. It is a graphical representation of an equilibrium state attainable by the system. Basically, there are three types of attractors. The simplest is the fixed point whereby the output of the system is a steady state. We showed an example of a fixed-point attractor earlier with the logistic map when the parameter value was set at $a = 2.5$. Despite repeated iterations, the system is attracted to the solution 0.60. The equilibrium is a single value.

The second type of attractor is the limit cycle whereby the trajectory repeats itself in a cyclic fashion. The logistic map also illustrated this process earlier. When $a = 3.2$, the system is attracted to a two-point equilibrium state in which the values alternate between 0.513045 and 0.799455. Similarly, when $a = 3.5$, the system is attracted to a different limit cycle consisting of a repeating sequence of four values: 0.826941, 0.500884, 0.874997, and 0.382820.

The third type of attractor is the strange attractor whereby the trajectory consists of aperiodic paths. The logistic map illustrated such a strange attractor. When $a = 3.9$, the system is attracted to infinite solutions all bounded within the range between 0 and 1. There are infinite equilibrium states, yet all are confined within a region of phase space.

The logistic map demonstrates that fixed points, various limit cycles and strange attractors can coexist within the same dynamical system. The system's behavior is dependent upon the parameter value.

One can use the data from a system to test whether a deterministic process exits. There are two approaches to the analysis of this data, the purposes of which are to gain additional descriptive insight about the dynamics of the system. The metric approach focuses on the distance between points on the attractor. The topological approach focuses on the organization of the attractor. Both approaches depend on the behaviors evidenced by the time-series data (Gilmore, 1993, 1996).

Kaminski and Wetzel (2004) used chaos theory to explore the dynamics of the financial accounting system, focusing on financial ratios of both fraudulent and non-fraudulent firms for evidence of non-linearity. Utilizing the metric approach, they employed four statistical tools to determine the behavior of the time-series. Their findings indicated that ratios comprised of financial data as reported on the balance sheet were the result of non-linear chaotic dynamics. None of the ratios they tested exhibited stable (i.e., fixed point) or periodic behavior (i.e., limit cycle).

The present study is an extension of Kaminski and Wetzel (2004), utilizing the topological approach. Through a phase space reconstruction of the shape of the attractor, additional descriptive insights about the dynamics of the system may be found. The phase space map enables a visible inspection of the attractor and provides evidence as to the type of attractor (i.e., fixed point, limit cycle, strange). According to Kellert (1993), reconstruction of attractors is one of the most important methods for discovering and analyzing chaos.

IV. Methodology

This research studies a longitudinal sample of financial statement data from fraudulent and non-fraudulent U.S. firms, using ratios computed from balance sheet and income statement data. This work extends the exploration conducted by Kaminski and Wetzel (2004) and is similarly exploratory in nature. However, the present research demonstrates the use of the topological approach. We note that qualitative analyses of phase space maps are by their nature subjective; thus, it is necessary to employ a data set where the underlying dynamics has already been ascertained. Given this need, the Kaminski and Wetzel (2004) data set was utilized, thereby eliminating confounding variables while also triangulating the findings of the prior study.

Kaminski and Wetzel (2004) give complete details of the data collection and we only summarize them here. We obtained all data for firms in this study from the SEC's Accounting and Auditing Enforcement Releases (AAERs) issued between 1982 and 1999. We identified a firm as "fraudulent" if the SEC accused top management of reporting materially false and misleading financial statements, thereby violating Rule 10(b)-5. Dalnial et al. (2014a) suggested "future research ... using other forms of data such as quarterly financial statements." (2014a, p. 68). That is precisely what the current study does, using quarterly financial statements for the entire time period for which SEC 10Q financial statements are available. We used a minimum time period of seven years (i.e., 28 quarters) inclusive of the "fraud year", defined as the first year for which the financial statements included fraudulent data. In many instances, the discovery of fraud occurred several years after the fraud year. Each fraud firm was matched with a non-fraud firm based on firm size, time period and industry. The firms may have had different pre-fraud and post-fraud periods but for each, the time-series method captured changes in the dynamics of the data.

Kaminski and Wetzel (2004) chose a parsimonious yet comprehensive and representative selection of financial ratios. For a thorough discussion of the considerations which governed the process of ratio selection, see Kaminski and Wetzel (2004). See Table 2 for a listing of the selected ratios. (See Table 2, pg. 11)

Topological analysis consists of constructing phase space maps and were created using the software program Chaos Data Analyzer—The Professional Version (Sprott, 1995) (CDA). This program has been used in several published studies, including those applying chaos methodology to financial data (e.g., Kaminski and Wetzel, 2004; Lindsay and Campbell, 1996). The researchers constructed a phase space map for each ratio for each fraud and non-fraud firm. The maps provide evidence of non-linear dynamics or the lack thereof in financial statement data and also indicate whether there are differences in the attractors of the various financial ratios and the attractors of the fraudulent versus non-fraudulent firms.

V. Results

For each of the 30 fraud and non-fraud firms, we computed the 10 financial ratios identified in Table 2 (i.e., R1 through R10) for the entire time period for which SEC 10Q financial statements were available. The minimum time period for inclusion in the study was seven years. This period was to allow a minimum time-series of 28 data points for analysis for each ratio. See Kaminski and Wetzel (2004) for descriptive statistics of the sample data. (See Figures 1–4, pgs. 13–14)

To determine the sensitivity of sample size on the phase space maps, preliminary tests and analyses were performed on known random, periodic and chaotic time-series. Figures 1 and 2 consist of phase space maps of a random time-series with a sample size of 200 data points. We created Figure 1 from the original time-series and Figure 2 from a random shuffle of the data. Figures 3 and 4 consist of similar maps but the sample size was only 50 data points. As evidenced by the figures, random data appears to fill the plane without any discernable pattern and produces similar maps for both the original and shuffled data. This result is true for both the large and small data sets. (See Figures 5–8, pgs. 15–16)

We constructed similar phase space maps for a periodic time-series and present them in Figures 5 through 8. The phase space map of the shuffled data is vastly different from that of the original data and closely resembles the maps of the random time-series. Such a map is a visual representation of how a shuffling of the data destroys the dynamics occurring within the original time-series. (See Figures 9–12, pgs. 17–18)

We present the phase space maps for a chaotic time-series in Figures 9 through 12. The map of the original data depicts a strange attractor with a discernable pattern. This pattern is still in evidence even with the small data set. The maps of the shuffled data are again vastly different and resemble the maps of the random time-series. We can conclude that, even for firms with a small number of data points (e.g., 28 quarters), the phase space maps depict the underlying dynamics of the time-series.

We constructed two phase space maps for each fraud and non-fraud firm for each of the ten ratios. One map was created using the original data. The second map was created from a random shuffling of the data. Given that this study had a total of 60 firms, each with ten ratios and each ratio having two maps, this procedure resulted in 1,200 maps requiring analysis.

We then classified the phase space maps for purposes of comparison. First, a phase space map of a random time-series will fill the plane without any discernable pattern. Similar findings result from a phase space mapping of the shuffled data. Second, a phase space mapping of a periodic or chaotic time-series will not fill the plane; the mapping depicts an attractor with a discernable pattern. Finally, shuffling the data of a periodic or chaotic time-series will destroy the original pattern and result in a map that fills the plane without any pattern.

A map was coded as "random" if the points comprising the map appeared to fill the plane and/or the shuffled map was similar to the original. An "**R**" was used to indicate a map that clearly appeared random, and a "r" was used for those maps that still appeared random but were less clearly discerned.

A map was coded as "chaotic" if there appeared to be some type of attractor or grouping of points and/or the shuffled map was vastly different from the original. Similarly, a "**C**" was used to indicate a map that clearly appeared chaotic, while a "c" was used for those maps that still appeared chaotic but were less clearly discerned. None of the maps indicated a periodic time-series, thereby eliminating the need for this classification.

The sample sizes ranged from 28 to 80 data points for the fraud firms and 30 to 89 data points for the non-fraud firms. The firms with the smaller sample sizes produced less definitive phase space maps and were generally coded with an "r" or a "c". Given the exploratory nature of this study and the small data sets employed, the above determinations were subjective and were based upon the preliminary analysis of the maps of the random, periodic and chaotic time-series described previously.

Ranked in order of decreasing randomness, an original map may be coded "**R**", "r", "c", or "**C**". The shuffle should destroy the dynamics within the system and result in a "random" code. A shuffled map was given a code of "**R**" only if the original map was clearly random and the shuffled map appeared identical. All other shuffled maps received a code of "r". The resulting possible classifications were "**RR**," "rr," "**C**r," and "cr". The first letter indicates the classification of the original map while the second letter indicates the classification of the shuffled map. For an "**RR**" classification, both the original map and the shuffled map have no discernable pattern and fill the plane. For an "rr" classification, both maps appear to be the result of a random time-series, though this conclusion is more subjective than that reached in the prior classification. A "**C**r" classification is when the original map appears to be of a strange attractor with a pattern or grouping of points thereby indicating a chaotic time-series. The shuffled map has no pattern and appears to fill the plane. A "cr" classification occurs when the original map appears to have a grouping of points but this conclusion is more subjective than that reached in the prior classification. The shuffled map has no pattern and appears random. (See Table 3, pg. 12)

We present a summary of the phase space map classifications in Table 3. We report percentages of each classification for each ratio, first for the fraud firm grouping (F), next for the non-fraud firm grouping (N) and finally for the total data set. The highest percentages of "**C**r" and "cr" classifications occur for R4, R7, and R10. R9 appears borderline. The phase space maps provide evidence of the type of attractor for each ratio for each of the fraud and non-fraud firms. R4, R7, and R10 appear to have strange attractors. R9 has similar though less conclusive results. These findings hold true for the fraud firm and non-fraud firm groupings as well as for the total data set.

VI. Conclusion

Summarizing the results of the phase space maps, we observed that the following ratios were consistently strong indicators of chaos: fixed assets to total assets (R4); total liabilities to total assets (R7); and current assets to current liabilities (R10). The ratio accounts receivable to inventory (R9) also identified a chaotic time-series but less consistently. Each of these ratios is comprised of balance sheet account(s) and/or categories and is the result of non-linear chaotic dynamics. The remaining tested ratios were comprised solely of income statement amounts or a combination of income statement and balance sheet amounts and did not exhibit chaos.

These findings were true for both firm types. Ratios cannot be used to differentiate fraud and non-fraud firms; this problem is consistent with prior research findings of the limited ability of financial ratios to detect fraudulent financial reporting (Alshehadeh and Atieh, 2020; Dalnial et al., 2014a; Guan et al., 2007; Kaminski and Wetzel, 2004; Kaminski et al., 2004; Persons, 1995). Beasley et al. (1999) found that fraudulent firms overstate asset accounts, particularly inventory, accounts receivable and fixed assets. Despite the use of these accounts/categories in nine of the 10 tested ratios, we found no difference in the underlying dynamics between fraud and non-fraud firms.

Overall, in this study we confirmed the existence of evidence of non-linearity in ratios comprised solely of balance sheet accounts/categories. The findings from this topological approach triangulated the findings using the metric approach discussed in Kaminski and Wetzel (2004). This study reached the same conclusion: " … ratios comprised of financial data as reported on quarterly balance sheets are the result of non-linear chaotic dynamics" (Kaminski and Wetzel, 2004, p. 169).

This study also provided further evidence that the exclusive use of linear models derived from financial ratios is inappropriate. Etheridge and Sriram (1993) discussed the implications of chaos theory and non-linear dynamics for accounting researchers. They suggested that ignoring the underlying dynamics of the system would result in selecting models that do not robustly represent the system. This problem may explain the low explanatory power and inconsistent results of prior studies using predictive models that include financial ratios.

As with all empirical investigations, there are limitations to this study. Since this study is an extension of Kaminski and Wetzel (2004) and utilizes the same data set, the limitations of the prior study are applicable. These limitations include the following: no comprehensive theory of chaotic phenomena, the need for data points over a long period of time, the potential misclassification of a non-fraud firm, and no acceptable theoretical foundation for the selection of financial ratios. Unique to the current study is the subjective nature of the classification of the phase space maps as random or chaotic and to what degree (i.e., **R**, r, **C**, c), thereby potentially limiting the resulting inferences. This limit is mitigated by finding results consistent with the prior study's more objective metric approach.

Subject to these limitations, the current study found evidence of non-linearity, underscoring Etheridge and Sriram's (1993) challenge. The utilization of select ratios comprised of quarterly balance sheet amounts for predictive purposes will have limited success, especially if they are components of a linear model.

The importance of developing fraud prediction models cannot be stated strongly enough. The proliferation of fraud occurrences, as well as their financial and global societal impact, is of such a magnitude that the ramifications are felt in every company category, size, and industry. The ability to build effective fraud prediction models using financial ratios would be an enormous step toward mitigating the quantity and impact of financial fraud. Additional investigation of such predictive models is warranted. The findings in this study can be used in subsequent research as a precursor to the selection and construction of a non-linear fraud detection and/or prediction model.

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Table 1: Logistic Map

	a = 2.5	*a* = 3.2	*a* = 3.5	*a* = 3.9	*a* = 3.900001	*a* = 4.2	*a* = 3.9
X_0	0.200000	0.200000	0.200000	0.200000	0.200000	0.200000	0.200001
X_1	0.400000	0.512000	0.560000	0.624000	0.624000	0.672000	0.624002
X_2	0.600000	0.799539	0.862400	0.915034	0.915034	0.925747	0.915031
X_3	0.600000	0.512884	0.415332	0.303214	0.303214	0.288705	0.303221
X_4	0.600000	0.799469	0.849910	0.823973	0.823973	0.862489	0.823984
X_5	0.600000	0.513019	0.446472	0.565661	0.565662	0.498128	0.565633
X_6	0.600000	0.799458	0.864971	0.958185	0.958186	1.049985	0.958200
X_7	0.600000	0.513040	0.408785	0.156258	0.156258	-0.220432	0.156206
X_8	0.600000	0.799456	0.845880	0.514181	0.514181	-1.129893	0.514042
X_9	0.600000	0.513044	0.456285	0.974216	0.974216	-10.107521	0.974231
X_{10}	0.600000	0.799456	0.868312	0.097966	0.097965	-471.531869	0.097909
X_{11}	0.600000	0.513044	0.400213	0.344638	0.344634	negative infinity	0.344460
X_{12}	0.600000	0.799455	0.840149	0.880864	0.880860	negative infinity	0.880648
X_{13}	0.600000	0.513044	0.470046	0.409276	0.409288	negative infinity	0.409917
X_{14}	0.600000	0.799455	0.871860	0.942900	0.942909	negative infinity	0.943352
X_{15}	0.600000	0.513045	0.391022	0.209975	0.209945	negative infinity	0.208412
X_{16}	0.600000	0.799455	0.833433	0.646953	0.646886	negative infinity	0.643409
X_{17}	0.600000	0.513045	0.485879	0.890778	0.890856	negative infinity	0.894793
X_{18}	0.600000	0.799455	0.874302	0.379440	0.379203	negative infinity	0.367142
X_{19}	0.600000	0.513045	0.384643	0.918314	0.918091	negative infinity	0.906160
X_{20}	0.600000	0.799455	0.828425	0.292551	0.293279	negative infinity	0.331634
X_{21}	0.600000	0.513045	0.497480	0.807164	0.808339	negative infinity	0.864446
X_{22}	0.600000	0.799455	0.874978	0.607036	0.604217	negative infinity	0.456999
X_{23}	0.600000	0.513045	0.382871	0.930319	0.932642	negative infinity	0.967789
X_{24}	0.600000	0.799455	0.826983	0.252821	0.245002	negative infinity	0.121578
X_{25}	0.600000	0.513045	0.500788	0.736720	0.721407	negative infinity	0.416508
X_{26}	0.600000	0.799455	0.874998	0.756458	0.783818	negative infinity	0.947813
X_{27}	0.600000	0.513045	0.382818	0.718494	0.660845	negative infinity	0.192906
X_{28}	0.600000	0.799455	0.826939	0.788815	0.874103	negative infinity	0.607205
X_{29}	0.600000	0.513045	0.500887	0.649684	0.429184	negative infinity	0.930178
X_{30}	0.600000	0.799455	0.874997	0.887619	0.955442	negative infinity	0.253293
X_{31}	0.600000	0.513045	0.382820	0.389030	0.166032	negative infinity	0.737629
X_{32}	0.600000	0.799455	0.826941	0.926974	0.540016	negative infinity	0.754777
X_{33}	0.600000	0.513045	0.500884	0.264003	0.968755	negative infinity	0.721845
X_{34}	0.600000	0.799455	0.874997	0.757791	0.118047	negative infinity	0.783061
X_{35}	0.600000	0.513045	0.382820	0.715820	0.406037	negative infinity	0.662519
X_{36}	0.600000	0.799455	0.826941	0.793345	0.940567	negative infinity	0.871992
X_{37}	0.600000	0.513045	0.500884	0.639400	0.218012	negative infinity	0.435325
X_{38}	0.600000	0.799455	0.874997	0.899214	0.664884	negative infinity	0.958687
X_{39}	0.600000	0.513045	0.382820	0.353451	0.868973	negative infinity	0.154464
X_{40}	0.600000	0.799455	0.826941	0.891241	0.444051	negative infinity	0.509360
X_{41}	0.600000	0.513045	0.500884	0.378028	0.962792	negative infinity	0.974658
X_{42}	0.600000	0.799455	0.874997	0.916979	0.139711	negative infinity	0.096328

X_{43}	0.600000	0.513045	0.382820	0.296901	0.468748	negative infinity	0.339491
X_{44}	0.600000	0.799455	0.826941	0.814129	0.971191	negative infinity	0.874523
X_{45}	0.600000	0.513045	0.500884	0.590161	0.109117	negative infinity	0.427956
X_{46}	0.600000	0.799455	0.874997	0.943297	0.379122	negative infinity	0.954758
X_{47}	0.600000	0.513045	0.382820	0.208602	0.918016	negative infinity	0.168462
X_{48}	0.600000	0.799455	0.826941	0.643840	0.293525	negative infinity	0.546321
X_{49}	0.600000	0.513045	0.500884	0.894309	0.808735	negative infinity	0.966632
X_{50}	0.600000	0.799455	0.874997	0.368628	0.603262	negative infinity	0.125793
X_{51}	0.600000	0.513045	0.382820	0.907692	0.933415	negative infinity	0.428880
X_{52}	0.600000	0.799455	0.826941	0.326771	0.242392	negative infinity	0.955274
X_{53}	0.600000	0.513045	0.500884	0.857968	0.716189	negative infinity	0.166631
X_{54}	0.600000	0.799455	0.874997	0.475251	0.792723	negative infinity	0.541575
X_{55}	0.600000	0.513045	0.382820	0.972611	0.640822	negative infinity	0.968259
X_{56}	0.600000	0.799455	0.826941	0.103891	0.897660	negative infinity	0.119861
X_{57}	0.600000	0.513045	0.500884	0.363081	0.358280	negative infinity	0.411427
X_{58}	0.600000	0.799455	0.874997	0.901887	0.896670	negative infinity	0.944404
X_{59}	0.600000	0.513045	0.382820	0.345098	0.361345	negative infinity	0.204770
X_{60}	0.600000	0.799455	0.826941	0.881421	0.900022	negative infinity	0.635073
X_{61}	0.600000	0.513045	0.500884	0.407620	0.350932	negative infinity	0.903845
X_{62}	0.600000	0.799455	0.874997	0.941717	0.888337	negative infinity	0.338945
X_{63}	0.600000	0.513045	0.382820	0.214054	0.386858	negative infinity	0.873839
X_{64}	0.600000	0.799455	0.826941	0.656117	0.925076	negative infinity	0.429953
X_{65}	0.600000	0.513045	0.500884	0.879947	0.270311	negative infinity	0.955865
X_{66}	0.600000	0.799455	0.874997	0.411997	0.769248	negative infinity	0.164531
X_{67}	0.600000	0.513045	0.382820	0.944796	0.692271	negative infinity	0.536097
X_{68}	0.600000	0.799455	0.826941	0.203410	0.830824	negative infinity	0.969918
X_{69}	0.600000	0.513045	0.500884	0.631934	0.548165	negative infinity	0.113789
X_{70}	0.600000	0.799455	0.874997	0.907114	0.965953	negative infinity	0.393281
X_{71}	0.600000	0.513045	0.382820	0.328607	0.128264	negative infinity	0.930583
X_{72}	0.600000	0.799455	0.826941	0.860435	0.436068	negative infinity	0.251933
X_{73}	0.600000	0.513045	0.500884	0.468337	0.959060	negative infinity	0.735005
X_{74}	0.600000	0.799455	0.874997	0.971090	0.153130	negative infinity	0.759613
X_{75}	0.600000	0.513045	0.382820	0.109489	0.505756	negative infinity	0.712144

Table 2: Financial Ratios

	Ratio	Classification
R1	Net Income / Total Assets	Return on Investment
R2	Net Income / Sales	
R3	Sales / Total Assets	Capital Intensiveness
R4	Fixed Assets / Total Assets	
R5	Inventory / Sales	Inventory Intensiveness
R6	Current Assets / Sales	
R7	Total Liabilities / Total Assets	Financial Leverage
R8	Sales / Accounts Receivable	Receivables Intensiveness
R9	Accounts Receivable / Inventory	
R10	Current Assets / Current Liabilities	Short-Term Liquidity

Table 3: Phase Space Maps—Percentages of Classification

Classification of Map Pattern										
	Financial Ratios									
	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10
Fraud Firms										
RR	0.9000	0.8667	0.6667	0.2667	0.6429	0.7333	0.1667	0.6000	0.4074	0.2333
rr	0.1000	0.1000	0.1333	0.1667	0.1429	0.2000	0.0667	0.2333	0.1852	0.3000
Cr			0.1000	0.3667	0.0357		0.3667		0.1481	0.0667
cr		0.0333	0.1000	0.2000	0.1786	0.0667	0.4000	0.1667	0.2593	0.4000
Non-Fraud Firms										
RR	0.8667	0.9667	0.8000	0.3333	0.6296	0.8000	0.2000	0.7667	0.5556	0.2333
rr	0.0667		0.0667	0.0667	0.1111	0.1333	0.0667	0.1667	0.1852	0.2000
Cr				0.3667	0.1111		0.4000		0.1481	0.3000
cr	0.0667	0.0333	0.1333	0.2333	0.1481	0.0667	0.3333	0.0667	0.1111	0.2667
All Firms										
RR	0.8833	0.9167	0.7333	0.3000	0.6364	0.7667	0.1833	0.6833	0.4815	0.2333
rr	0.0833	0.0500	0.1000	0.1167	0.1273	0.1667	0.0667	0.2000	0.1852	0.2500
Cr			0.0500	0.3667	0.0727		0.3833		0.1481	0.1833
cr	0.0333	0.0333	0.1167	0.2167	0.1636	0.0667	0.3667	0.1167	0.1852	0.3333

Figure 1: Random Time Series, Original Data Set (n = 200)

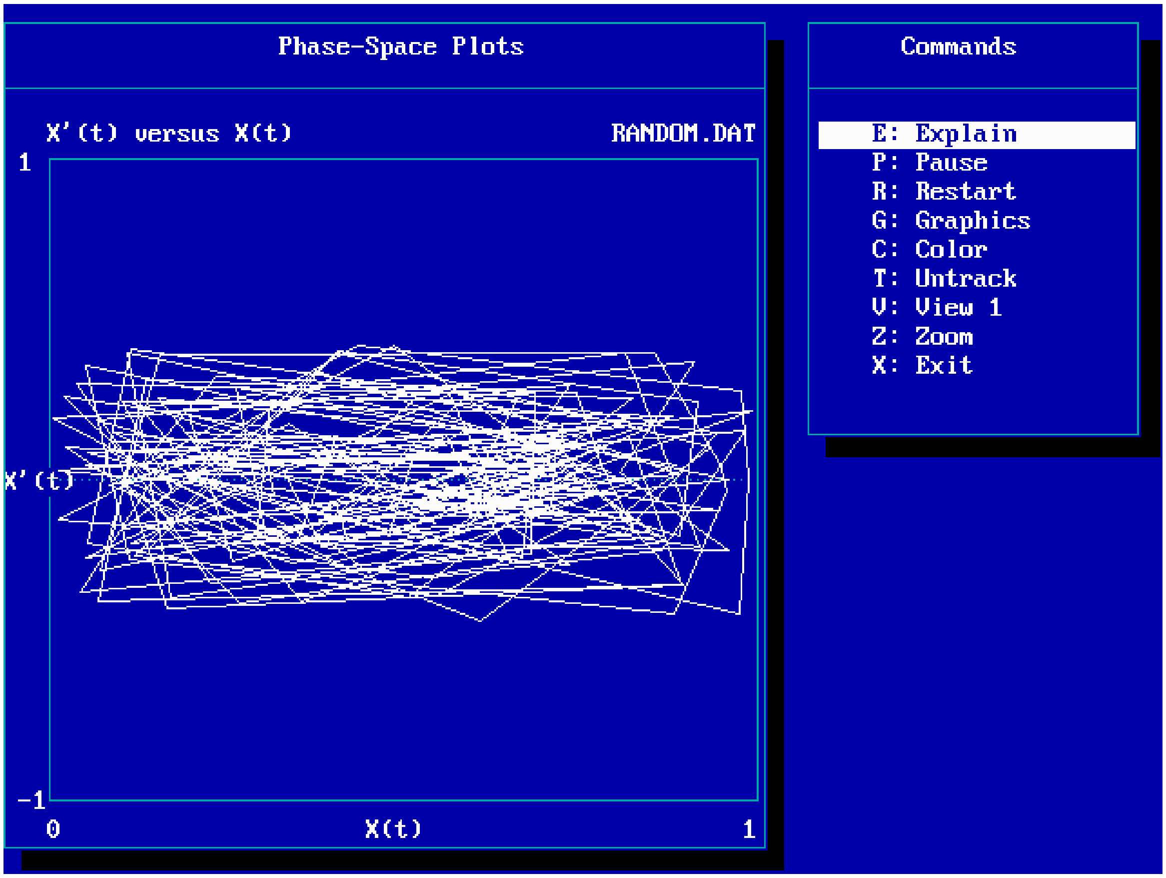

Figure 2: Random Time Series, Shuffled Data Set (n = 200)

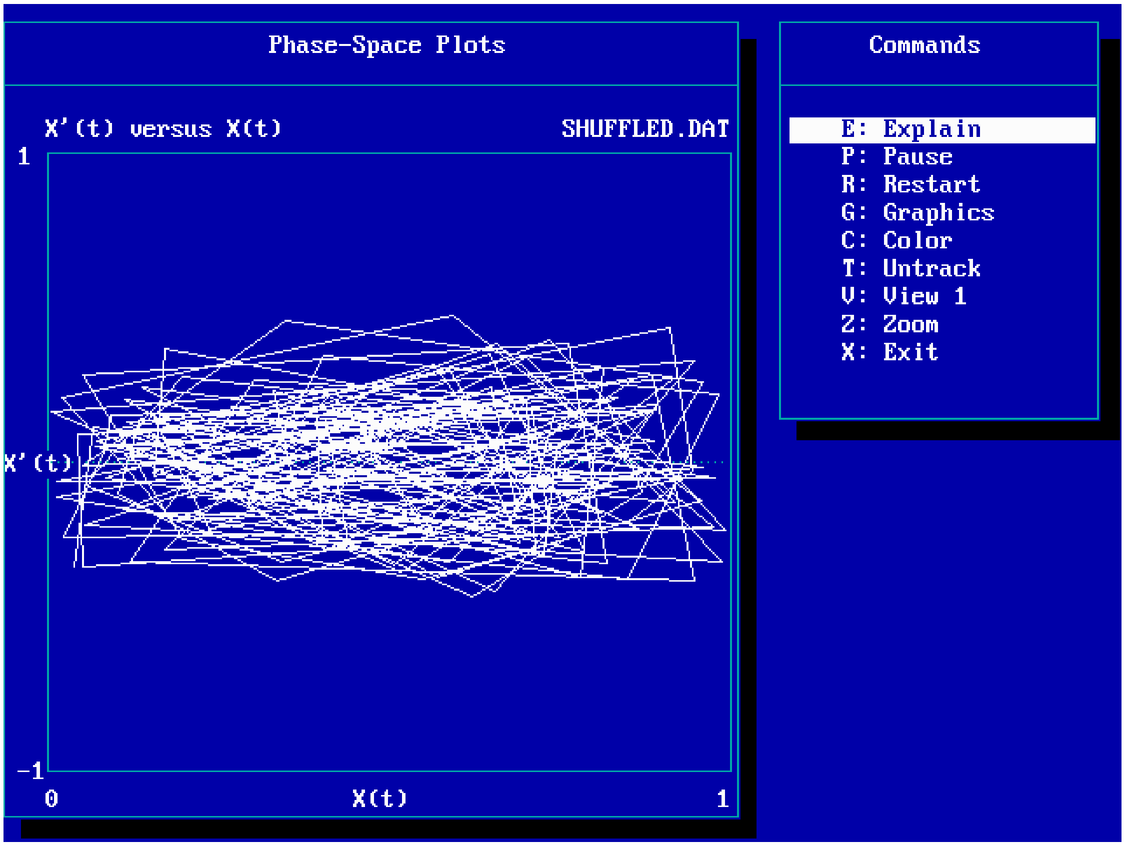

Figure 3: Random Time Series, Original Data Set (n =50)

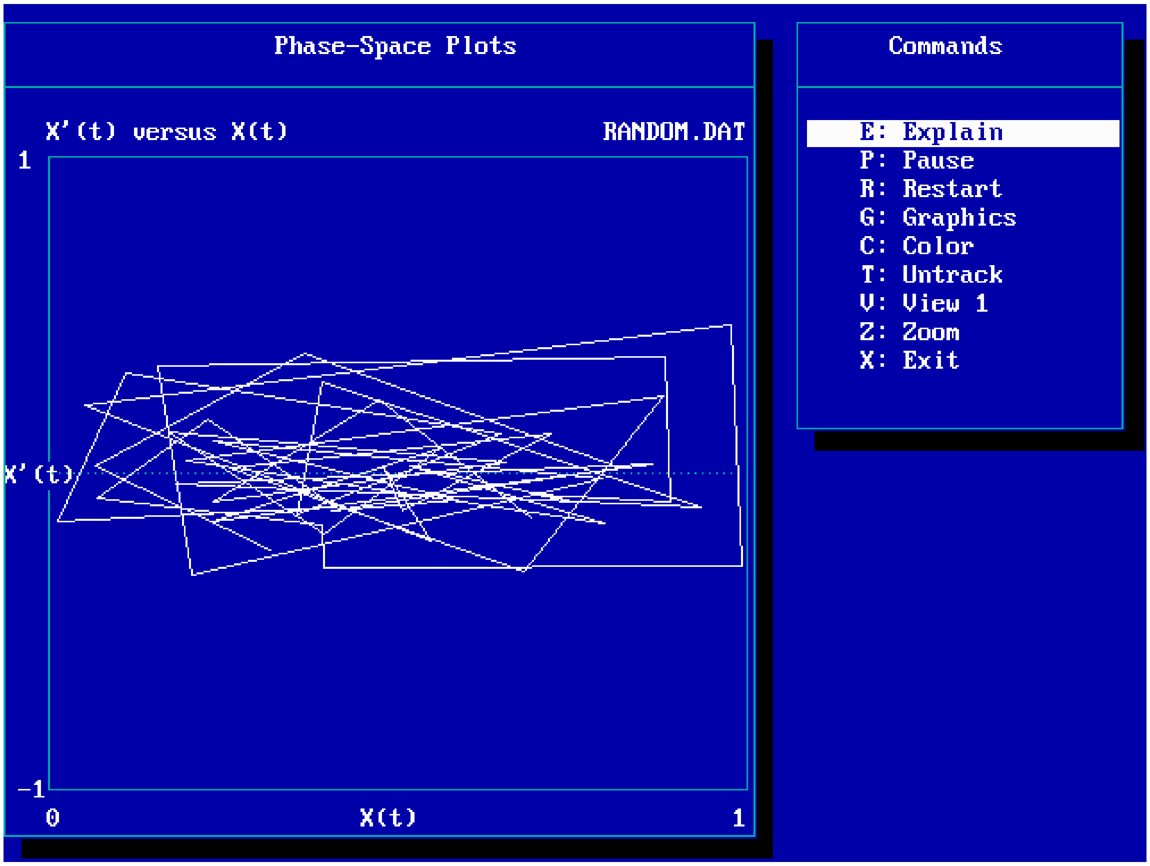

Figure 4: Random Time Series, Shuffled Data Set (n = 50)

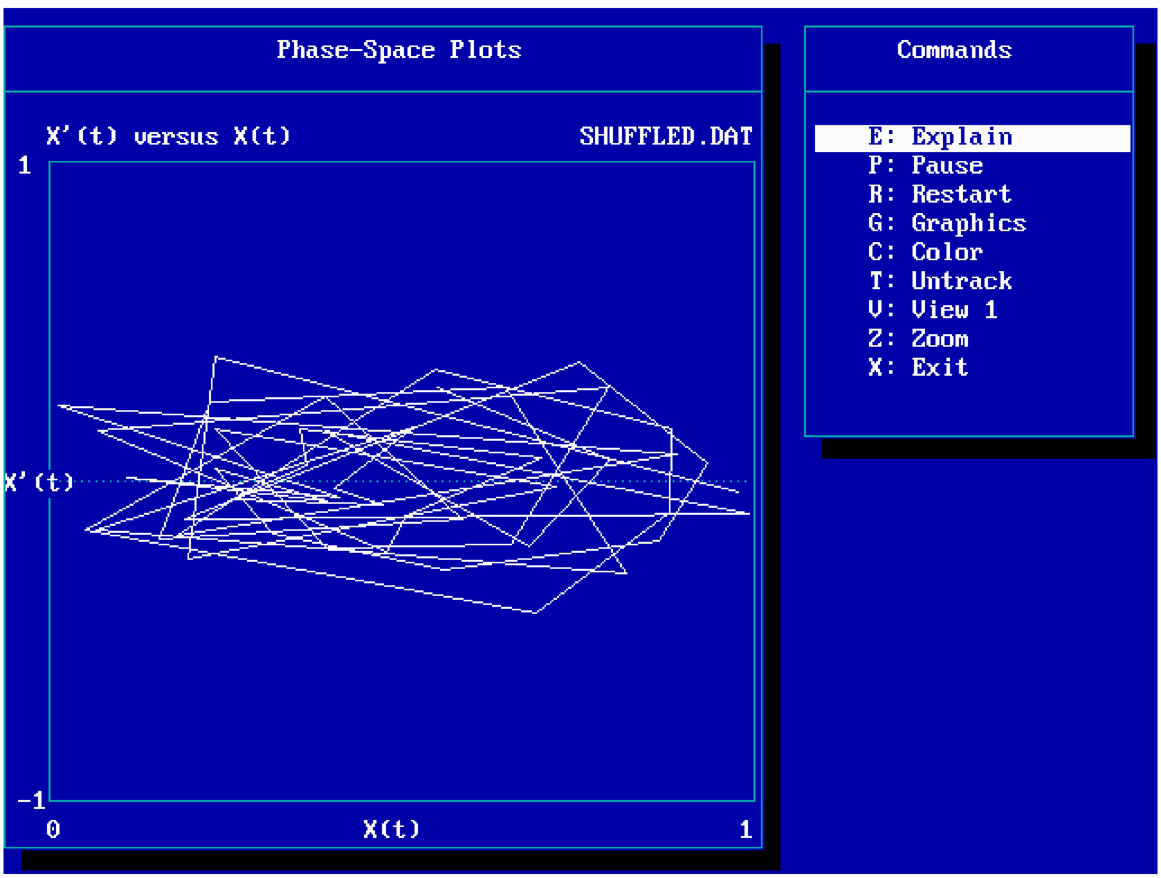

Figure 5: Periodic Time Series, Original Data Set (n = 200)

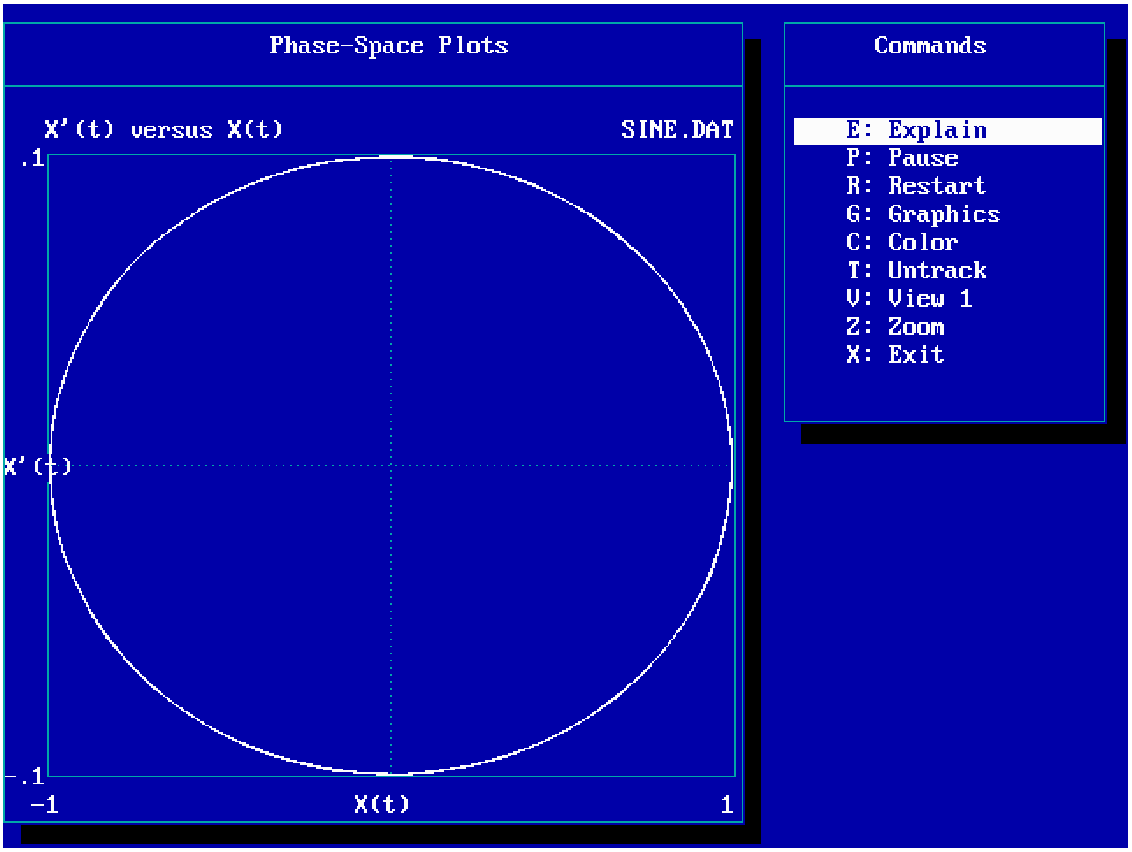

Figure 6: Periodic Time Series, Shuffled Data Set (n = 200)

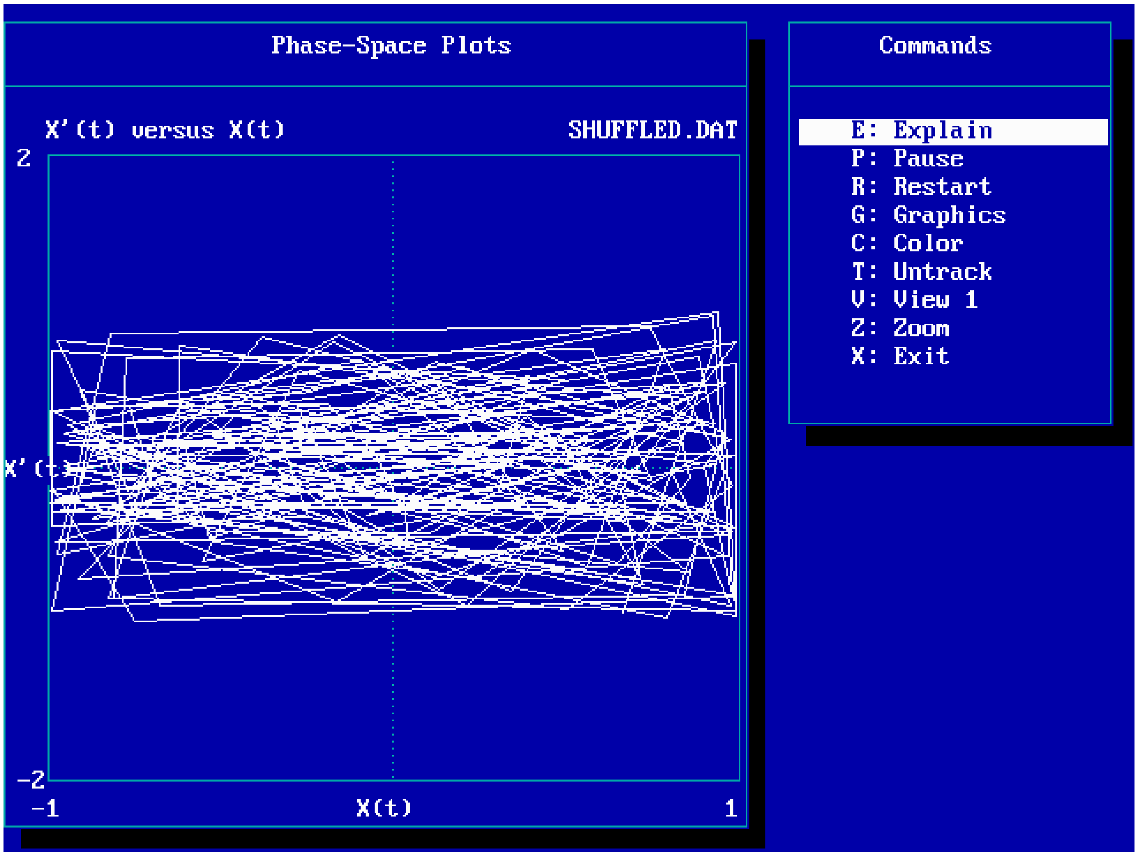

Figure 7: Periodic Time Series, Original Data Set (n = 50)

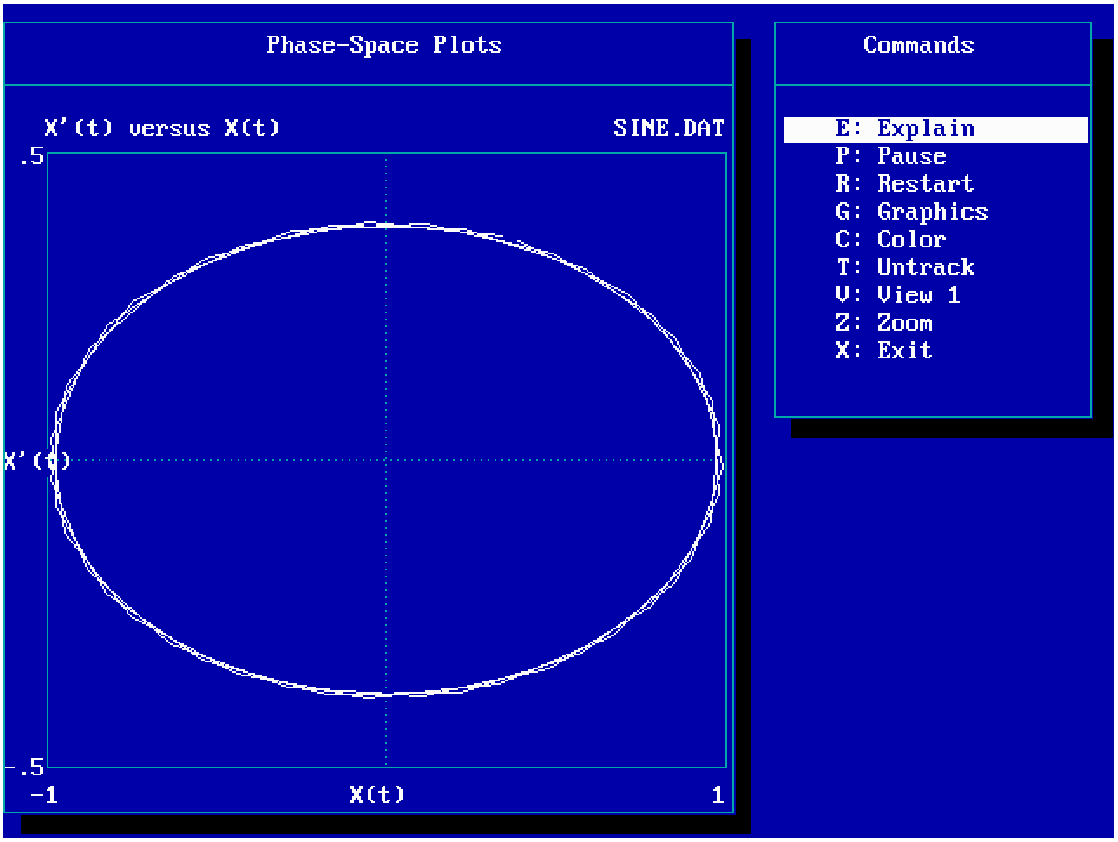

Figure 8: Periodic Time Series, Shuffled Data Set (n = 50)

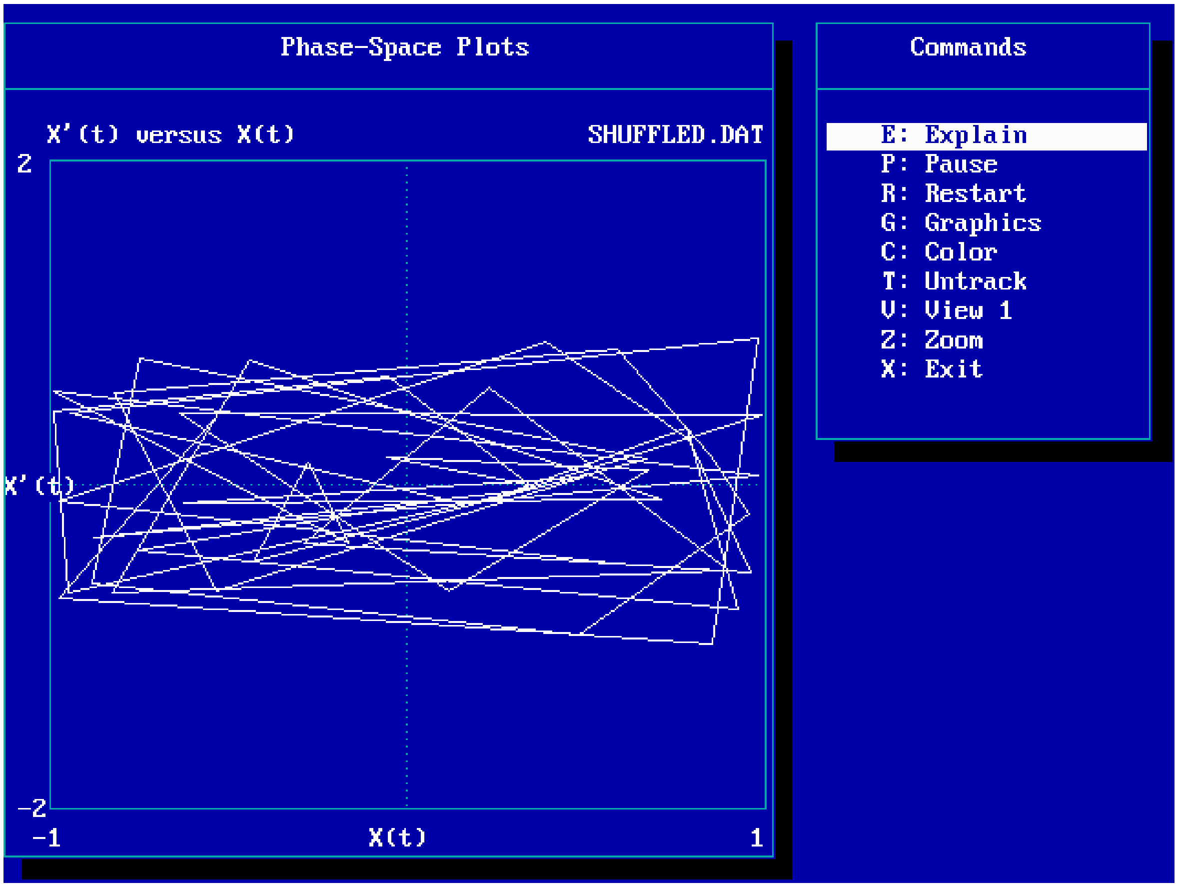

Figure 9: Chaotic Time Series, Original Data Set (n = 200)

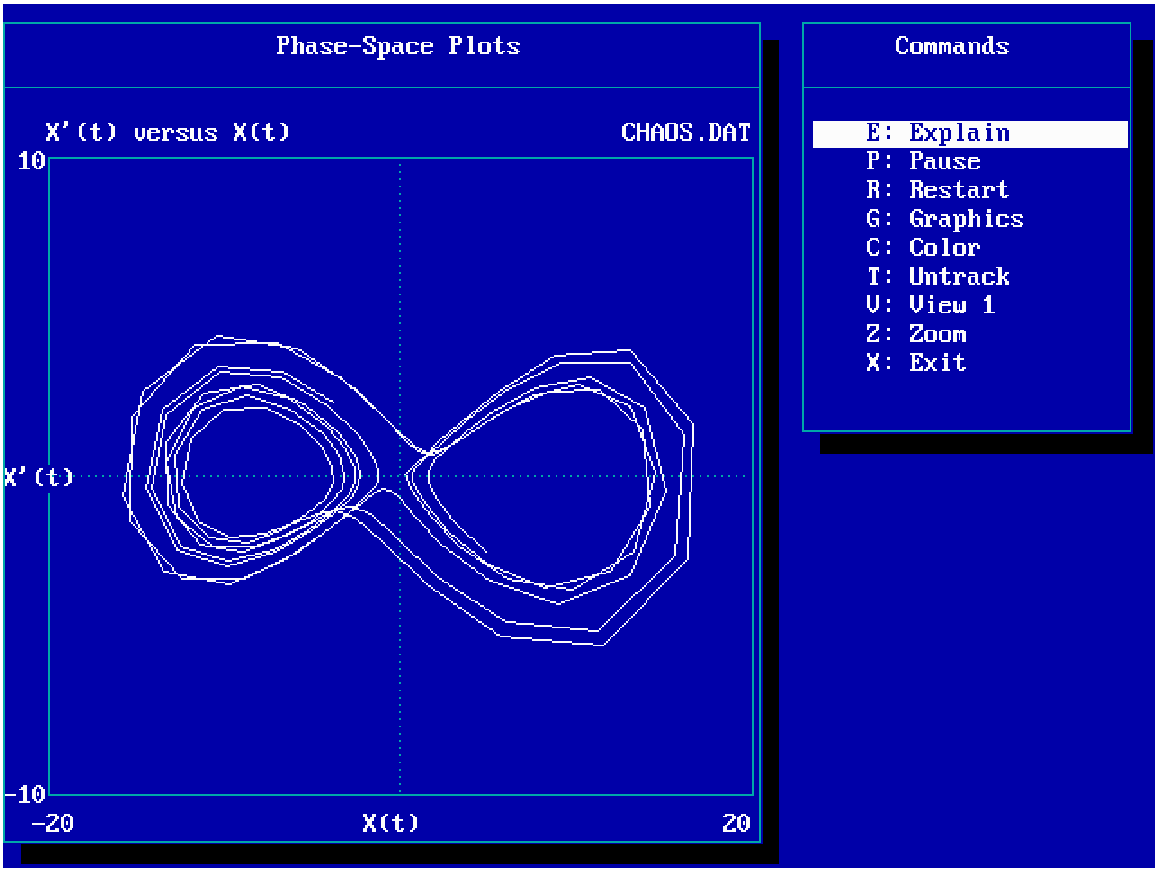

Figure 10: Chaotic Time Series, Shuffled Data Set (n = 200)

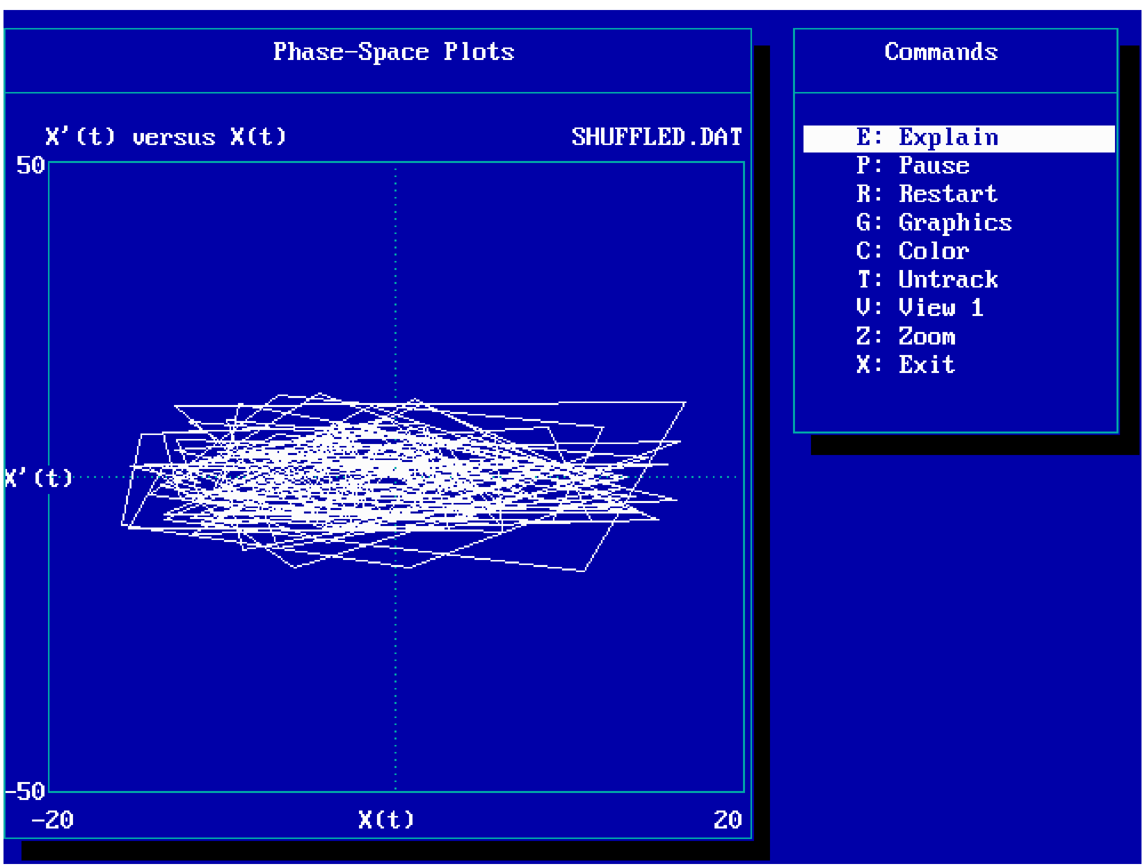

Figure 11: Chaotic Time Series, Original Data Set (n = 50)

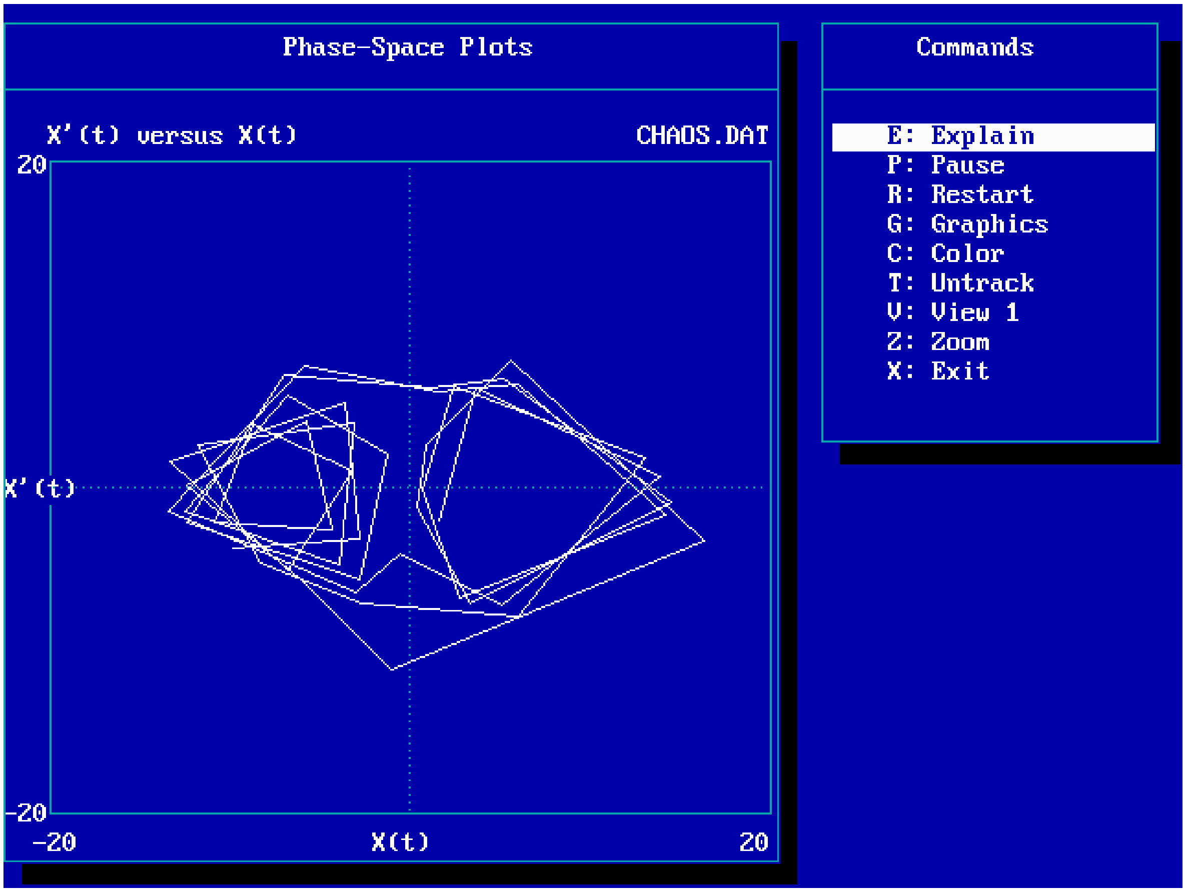

Figure 12: Chaotic Time Series, Shuffled Data Set (n = 50)